The Characteristic Impedance of a Slotted Coaxial Line

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Summary—**The propagation of a second type of TEM mode in a slotted coaxial line is analyzed. The characteristic impedance of the slotted line is evaluated by means of variational expressions giving upper and lower bounds to the true value. A two term approximation to the charge distribution and a one term approximation to potential distribution give results accurate to within ±2 per cent. Curves of characteristic impedance against angular slot width are presented.**

Introduction

IN A SLOTTED coaxial line as illustrated in Fig. 1 two types of TEM modes may propagate. The first is a perturbed fundamental coaxial line mode. If the slots are narrow, then apart from a small amount of fringing of the field in the region of the slots, the field is confined entirely to the region between the inner and outer conductors. The second type is a mode with the electric field lines of force crossing the symmetry plane everywhere at right angles. Unlike the first type of TEM mode, the field of this second mode extends into all of the space surrounding the coaxial line. The existence of this mode is of fundamental importance in the slotted coaxial line dipole antenna feed.[1] It is this mode that excites currents on the dipole wings and causes the dipole to radiate. This mode should not be confused with a perturbed second order coaxial line mode and in practice is usually excited by short circuiting inner conductor to outer conductor by a short circuiting post located in plane perpendicular to symmetry plane.

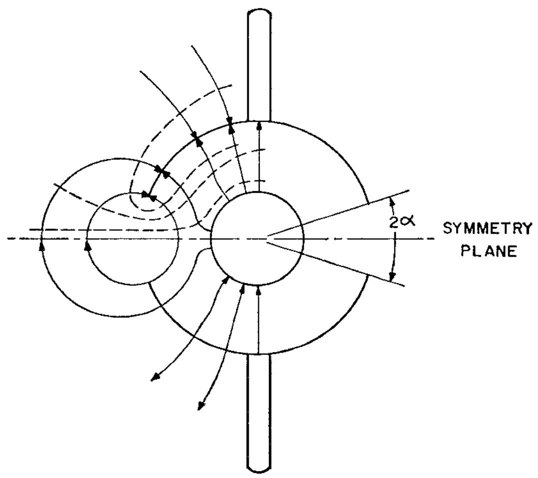

Fig. 1—Field distribution of second type of TEM mode illustrating excitation of dipole wings. —— electric field. ------ magnetic field.

This paper is primarily concerned with the evaluation of the characteristic impedance of the slotted coaxial line for this second type of transmission line mode. A rigorous solution in closed form has not been found. However by a variational method similar to that used in waveguide problems[2] upper and lower bounds to the characteristic impedance have been obtained. The procedure used is capable of giving a result of as high a degree of accuracy as desired. The upper bound to the characteristic impedance is found from a variational expression involving the charge on the outer conductor while the lower bound is obtained from a variational expression involving the potential distribution in the slotted regions. Due to the symmetry of the problem it is only necessary to consider the solution to the reduced problem illustrated in Fig. 2 and consisting of the two half coaxial cylinders above the symmetry plane and an infinite conducting plane placed coincident with the symmetry plane. The field distribution below the symmetry plane is the mirror image of that above. For this reason the current flowing on the lower half coaxial cylinders will be directed oppositely to that flowing on the upper half cylinders. There will be no net current flow on the center conductor; the current flowing down the line on one outer half cylinder and back on the other. It is seen that the slotted coaxial line is thus a balanced three wire line as far as this second type of TEM mode is concerned.

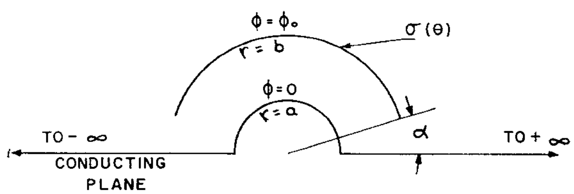

Fig. 2—Reduced problem obtained by image theory.

Upper Bound to Characteristic Impedance

For a TEM mode the field distribution in the transverse plane is a solution of Laplace's equation and may therefore be derived from an appropriate potential or stream function. In the cylindrical coordinate system r, θ, z the transverse electric and magnetic field components are related as follows:[3]

$$-j\omega\mu H_r = kE_\theta, \qquad j\omega\mu H_\theta = kE_r,$$

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[1] S. Silver, "Microwave antenna theory and design," McGraw-Hill Bk. Co., New York, 1949. Chapter 8, Sec. 4.

[2] J. W. Miles, "The equivalent circuit for a plane discontinuity in a cylindrical waveguide," Proc. IRE, Vol. 34, p. 728; October, 1946.

[3] MKS units are used and the time factor $e^{j\omega t}$ is dropped for convenience.

where

$$k = j\omega\sqrt{\mu\epsilon} = \frac{j2\pi}{\lambda}$$

The inner and outer radii of the coaxial cylinder are taken as "a" and "b" respectively and the angular slot width as 2α. If $\phi(r, \theta)$ is the potential distribution in the space surrounding the half cylinders when the outer half cylinder is held at a potential ϕ_0 with respect to the inner half cylinder and ground plane, then the electric field may be found from the following relation:

$$E = -\operatorname{grad}\phi$$

In order that the outer half cylinder shall be at a potential ϕ_0 there must be a distribution of charge $\sigma(\theta)$ on the half cylinder. This charge distribution is proportional to the discontinuity of the normal electric field at $r = b$ and hence is proportional to the discontinuity in the tangential magnetic field at $r = b$ and thus proportional to current flowing on line. Explicitly one has:

$$I = \sqrt{\frac{\epsilon}{\mu}}\left[\int_{\alpha}^{\pi-\alpha}\left(\frac{\partial\phi}{\partial r}\bigg|_{r=b-} - \frac{\partial\phi}{\partial r}\bigg|_{r=b+}\right)d\theta\right]$$
$$= \frac{1}{\sqrt{\mu\epsilon}}\int_{\alpha}^{\pi-\alpha}\sigma(\theta)d\theta = \frac{Q}{\sqrt{\mu\epsilon}}$$

where I is the total current flowing and Q is the total charge on the upper half cylinder.

The characteristic impedance Z_0 measured between the two outer half cylinders is $2\phi_0/I$ and is equal to

$$\frac{2\phi_0}{Q}\sqrt{\mu\epsilon} = \frac{1}{Cv}$$

where C is the total capacity of the slotted coaxial line per unit length and v is the velocity of light in the surrounding medium. Thus it suffices to determine the capacity C in order to evaluate Z_0.

Consider first of all solution of following equation:

$$\nabla^2 G(r, \theta, \theta') = -\frac{1}{\epsilon}\delta(r - b)\delta(\theta - \theta') \qquad (1)$$

which defines the Green's function G for the above problem. The delta functions have the property that $\int\delta(r-b)dr$ equals unity if the interval of integration includes the point b and equals zero otherwise. The Green's function is subject to the boundary conditions that it vanishes on the inner half cylinder and ground plane, is continuous with a discontinuous normal derivative at $r = b$, and is regular at infinity. The Green's function is thus seen to be the potential due to unit charge located at b, θ'. A suitable form for Green's function is readily found by standard methods,[4] giving

$$G(r, \theta, \theta') = \frac{2}{\epsilon\pi}\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{\sinh n \ln \frac{r}{a}\sin n\theta \sin n\theta'}{n\left(\sinh n \ln \frac{b}{a} + \cosh n \ln \frac{b}{a}\right)}, & r \leq b \\ \displaystyle\sum_{n=1}^{\infty}\frac{\sinh n \ln \frac{b}{a}\, e^{-n \ln r/b}\sin n\theta \sin n\theta'}{n\left(\sinh n \ln \frac{b}{a} + \cosh n \ln \frac{b}{a}\right)}, & r \geq b\end{cases} \qquad (2)$$

By the superposition theorem the potential due to a charge distribution $\sigma(\theta)$ at $r = b$ is:

$$\phi(r, \theta) = \int_{\alpha}^{\pi-\alpha}\sigma(\theta')G(r, \theta, \theta')d\theta' \qquad (3)$$

Imposing the boundary condition that $\phi = \phi_0$ on the outer half cylinder gives the following integral equation whose solution determines the charge distribution $\sigma(\theta)$:

$$\phi_0 = \int_{\alpha}^{\pi-\alpha}\sigma(\theta')G(b, \theta, \theta')d\theta'. \qquad (4)$$

To obtain a variational expression for Z_0 multiply the above integral equation by $\sigma(\theta)$ and integrate over $\alpha \leqq \theta \leqq \pi - \alpha$. Introducing the value of Z_0 given by

$$\frac{2\phi_0}{v\int_{\alpha}^{\pi-\alpha}\sigma(\theta)d\theta}$$

and dividing both sides by $[\int_{\alpha}^{\pi-\alpha}\sigma(\theta)d\theta]^2$ gives:

$$Z_0 = \frac{\frac{2}{v}\int\int_{\alpha}^{\pi-\alpha}G(b, \theta, \theta')\sigma(\theta)\sigma(\theta')d\theta d\theta'}{\left[\int_{\alpha}^{\pi-\alpha}\sigma(\theta)d\theta\right]^2} \qquad (5)$$

The value of Z_0 as given by this expression is easily shown to be stationary with respect to arbitrary first order variations in the charge distribution $\sigma(\theta)$ and hence is the required variational expression. Furthermore this expression is a positive definite quadratic form so the stationary value is an absolute minimum for the correct form of $\sigma(\theta)$ and will therefore give an upper bound for the impedance Z_0. A suitable set of functions in which to expand $\sigma(\theta)$ that will converge to the rigorously correct solution are the following cosine functions:

$$\sum_{s=0}^{S} c_s \cos\frac{2\pi s(\theta - \alpha)}{\pi - 2\alpha}.$$

The stationary value of Z_0 is obtained by substituting a finite number of terms of the above series into the variational expression, setting all the partial derivatives $\partial/\partial c_s$ equal to zero, solving the resultant equations for the c_s and finally substituting back into the variational expression and evaluating Z_0. There is no loss in gener-

[4] P. M. Morse and H. Feshback, *Methods of theoretical physics*, McGraw-Hill Book Co., New York, 1953.

ality by taking c_0 equal to unity since Z_0 depends only on the functional form of $\sigma(\theta)$. Using constant term and one cosine term following result is obtained for Z_0:

$$Z_0 = \frac{1920}{(\pi-2\alpha)^2} \sum_{n=1,3\cdots}^{\infty} \frac{\cos^2 n\alpha \left[1 + c_1 \frac{n^2(\pi-2\alpha)^2}{n^2(\pi-2\alpha)^2 - 4\pi^2}\right]^2}{n^3\left[1 + \coth n \ln \frac{b}{a}\right]} \text{ ohms} \tag{6}$$

where

$$c_1 = -\frac{\displaystyle\sum_{n=1,3\cdots}^{\infty} \frac{\cos^2 n\alpha}{n\left(1+\coth n \ln \frac{b}{a}\right)[n^2(\pi-2\alpha)^2 - 4\pi^2]}}{\displaystyle\sum_{n=1,3\cdots}^{\infty} \frac{[n\cos^2 n\alpha][\pi-2\alpha]^2}{\left(1+\coth n\ln\frac{b}{a}\right)[n^2(\pi-2\alpha)^2-4\pi^2]^2}}$$

The above series are quite suitable for numerical computation as the terms decrease as n^3 and the summations are over odd integers only.

Lower Bound to the Characteristic Impedance

In order to obtain an estimate of the error involved in calculating Z_0 *from the* previous expression it is necessary to develop an alternative expression that will give a lower bound to Z_0. This may be done by solving the original problem in terms of the potential distribution in the slots. A suitable expansion of the potential function is the following:

$$\phi(r,\theta) = \begin{cases} \displaystyle\sum_{n=1,3\cdots}^{\infty} a_n \sin n\theta \sinh n \ln \frac{r}{a}, & r \le b \\ \displaystyle\sum_{n=1,3\cdots}^{\infty} a_n \sinh n \ln \frac{b}{a} \sin n\theta e^{-n \ln r/b}, & r \ge b \end{cases} \tag{7}$$

If the potential in the region $0 \leqq \theta \leqq \alpha$, $r=b$ were known, then the coefficients a_n could be uniquely determined by Fourier analysis. However since the capacity C is required it will be sufficient to obtain a variational expression involving this unknown potential distribution $\phi(\theta)$ and which gives C as a stationary quantity. Since the electrostatic energy stored in the field is known to be a minimum and also proportional to $\phi_0^2 C$ it is clear that a variational expression for C can be developed by calculating the electrostatic energy in the field. The electric energy is given by:

$$W_e = \frac{\epsilon}{2}\int E^2 dv = \frac{\epsilon}{2}\int (\text{grad } \phi)^2 dv$$

Substituting for ϕ from (7), taking the gradient and integrating over the whole r, θ plane gives:

$$\frac{\pi\epsilon}{2}\sum_{n=1,3}^{\infty} a_n^2 n \sinh n \ln \frac{b}{a}\left(\cosh n \ln \frac{b}{a} + \sinh n \ln \frac{b}{a}\right) \tag{8}$$

for the total energy stored in the field per unit length of line. Now $2C\phi_0^2 = W_e$ and hence $Z_0 = \phi_0^2/vW_e$. The coefficients a_n are given by Fourier analysis as follows:

$$a_n \frac{\pi}{2}\sinh n \ln \frac{b}{a} = \int_0^\pi \phi \sin n\theta d\theta$$
$$= 2\left[\int_0^\alpha \phi \sin n\theta d\theta + \int_\alpha^{\pi/2} \phi_0 \sin n\theta d\theta\right]$$

since the potential reduces to ϕ_0 on the outer cylinder. Substituting into the expression for W_e gives:

$$Z_0^{-1} = \frac{4v\epsilon}{\pi\phi_0^2}\sum_{n=1,3\cdots}^{\infty} n\left(1+\coth n \ln \frac{b}{a}\right) \cdot \left(\int_0^{\pi/2} \phi \sin n\theta d\theta\right)^2. \tag{9}$$

To show that $1/Z_0$ is stationary with respect to arbitrary first order variations in the functional form of ϕ, consider the variation in W_e due to a variation $\delta\phi$ in ϕ, thus:

$$\delta W_e = \frac{8\epsilon}{\pi}\int_0^{\pi/2} \delta\phi \left[\sum_{n=1,3\cdots}^{\infty} n\left(1+\coth n\ln\frac{b}{a}\right) \sin n\theta \int_0^{\pi/2} \phi \sin n\theta d\theta\right] d\theta.$$

Since $\phi = \phi_0$ for $\alpha \leqq \theta \leqq \pi - \alpha$ the above result reduces to:

$$\delta W_e = \frac{8\epsilon}{\pi}\int_0^{\alpha} \delta\phi \left[\sum_{n=1,3\cdots}^{\infty} n\left(1+\coth n\ln\frac{b}{a}\right) \sin n\theta \int_0^{\pi/2} \phi \sin n\theta d\theta\right] d\theta.$$

For the first variation in δW_e to vanish

$$\sum_{n=1,3\cdots}^{\infty} n\left(1+\coth n \ln \frac{b}{a}\right)\sin n\theta \int_0^{\pi/2} \phi \sin n\theta d\theta$$

must equal zero over the region of the slots. This equation is also the condition that must be imposed on ϕ in order that the normal electric field in the slots be continuous and hence δW_e is identically zero. The expression for W_e is a positive definite quadratic form and therefore yields a lower bound to the characteristic impedance Z_0. A suitable form for the potential distribution that will result in a closed form for Z_0 is

$$\phi(\theta, b) = \begin{cases} \phi_0 \dfrac{\theta}{\alpha}, & 0 \leqq \theta \leqq \alpha. \\ \phi_0, & \alpha \leqq \theta \leqq \dfrac{\pi}{2}. \end{cases} \quad (10)$$

Substituting in the expression for Z_0 and performing the integration gives

$$\frac{1}{Z_0} = \frac{.00338}{\alpha^2} \sum_{1,3\cdots}^{\infty} \left(1 + \coth n \ln \frac{b}{a}\right) \frac{\sin^2 n\alpha}{n^3} \text{ ohms.} \quad (11)$$

The summation of the above series is carried out in the Appendix and gives finally

$$Z_0 = \frac{296 \text{ ohms}}{1.5 - \ln \alpha + \sum_{1,3\cdots}^{\infty} \left(\coth n \ln \frac{b}{a} - 1\right) \frac{\sin^2 n\alpha}{n^3 \alpha^2}}. \quad (12)$$

The series converges rapidly since coth n ln b/a approaches unity rapidly. Higher order approximations to Z_0 may be obtained with the following series expansion for $\phi(\theta)$,

$$\phi(\theta) = \phi_0 \left[\frac{\theta}{\alpha} + \sum_{s=1}^{\infty} c_s \sin \frac{s\pi\theta}{\alpha}\right] \text{ for } 0 \leqq \theta \leqq \alpha.$$

Numerical Example

Upper and lower bounds to the characteristic impedance Z_0 as obtained from (6) and (12) have been evaluated for the particular case of $b/a = 2.6$. Curves of Z_0 vs slot angle 2α are plotted in Fig. 3 for a range of 2α from 0° to 30°. The average values of Z_0 as obtained from (6) and (12) are also given. The value of Z_0 for $2\alpha = 20°$ was also evaluated by using a three term approximation to the upper bound and a two term approximation to the lower bound. This shows that the average value of Z_0 given in Fig. 3 is probably accurate to within ±2 per cent for most of the range of α considered. Characteristic impedance is relatively independent of ratio of b/a, provided ratio is greater than 3. In Fig. 4 average value of characteristic impedance is plotted as a function of slot angle for the case when ratio of outer to inner radii is large; i.e., $(b/a) > 6$. This value of Z_0 is average value obtained from (6) and (12) with coth n ln b/a replaced by unity.

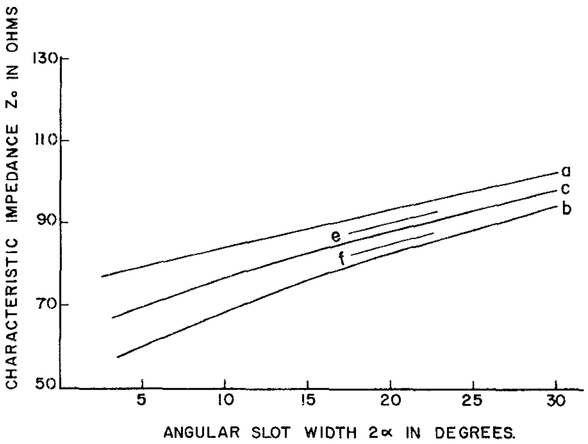

Fig. 3—Variation of characteristic impedance with slot angle for $b/a = 2.6$. (a) Two term approximation to upper bound. (b) One term approximation to lower bound. (c) Average of (a) and (b). (e) Three term approximation to upper bound. (f) Two term approximation to lower bound.

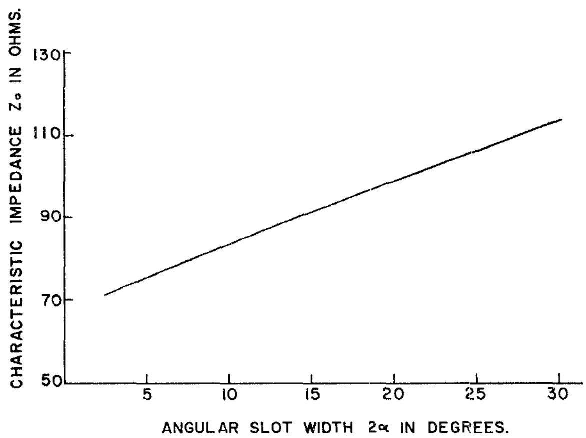

Fig. 4—Variation of characteristic impedance with slot angle for $b/a > 6$.

A plot of the charge distribution obtained for the three term approximation to the upper bound for Z_0 shows that the charge, and hence the current, is very heavily concentrated in the region of the slots.

Conclusions

The propagation of a second type of TEM mode on a slotted coaxial line has been discussed. In particular variational expressions giving upper and lower bounds to the characteristic impedance of the slotted line have been derived. The procedure developed is applicable generally and may be used to evaluate characteristic impedance of other structures occurring in practice.

For a generalized cylindrical transmission line the equivalent electrostatic problem is two dimensional. Let C_0 and C_1 be two open or closed curves coincident with the conducting surfaces of the line. Let $G(r, r')$ be the Green's function for the space surrounding C_0 in the absence of C_1 and such that G vanishes on C_0; i.e., G is the potential due to a unit charge at r' and such that the potential vanishes on C_0. The upper bound to the characteristic impedance of the line is then given by

$$Z_0 = \frac{\iint_{c_1} G(r, r')\sigma(r)\sigma(r') dr dr'}{v\left(\int_{c_1} \sigma(r) dr\right)^2}$$

where $\sigma(r)$ is the charge distribution on C_1. The lower bound to Z_0 may be found by minimizing the volume integral of the electrostatic energy density of the field.

Appendix

The series to be summed is

$$\sum_{n=1,3\cdots}^{\infty}\left(1+\coth n\ln\frac{b}{a}\right)\frac{\sin^2 n\alpha}{n^3}.$$

This series may be written as:

$$2\sum_{1,3\cdots}^{\infty}\frac{\sin^2 n\alpha}{n^3}+\sum_{1,3\cdots}^{\infty}\left(\coth n\ln\frac{b}{a}-1\right)\frac{\sin^2 n\alpha}{n^3}$$

where the second series is rapidly converging and in the form of a correction term. Integrating the well known geometric series

$$\sum_{1}^{\infty}e^{2jn\alpha}=\frac{e^{2j\alpha}}{1-e^{2j\alpha}}$$

once with respect to α and taking the real part gives

$$\sum_{1}^{\infty}\frac{\cos 2n\alpha}{n}=-\ln 2\sin\alpha.$$

Replacing 2α by $\pi-2\alpha$ and adding and subtracting series, it is readily deduced that

$$2\sum_{1,3\cdots}^{\infty}\frac{\cos 2n\alpha}{n}=-\ln\tan\alpha.$$

Integrating this series twice with respect to α gives

$$\sum_{1,3}^{\infty}\frac{\sin^2 n\alpha}{n^3}=\int_0^{\alpha}\int_0^{\tau}\ln\tan y\,dy\,dx.$$

For a limited range of α, ln tan y may be replaced by the first few terms of its Maclaurin's expansion and the integration may then be performed. One has

$$\ln\tan y=\ln y+\frac{y^2}{3}+\frac{7y^4}{90}+\cdots.$$

The integration gives

$$\sum_{1,3\cdots}^{\infty}\frac{\sin^2 n\alpha}{n^3}=(1.5-\ln\alpha)\frac{\alpha^2}{2}-\frac{\alpha^4}{36}-\cdots.$$

For $\alpha\leqq .5$ the following result is obtained:

$$\sum_{1,3\cdots}^{\infty}\left(1+\coth n\ln\frac{b}{a}\right)\frac{\sin^2 n\alpha}{n^3}$$
$$=(1.5-\ln\alpha)\alpha^2+\sum_{1,3}^{\infty}\left(\coth n\ln\frac{b}{a}-1\right)\frac{\sin^2 n\alpha}{n^3}.$$

Broadband Ferrite Microwave Isolator*

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Summary—A new type broadband unidirectional transmission line has been built utilizing the difference in energy distribution between two counter-rotating circularly polarized waves in a circular waveguide containing a ferrite. This principle of isolation is different from those which have been used previously.

A large difference is observed in the energy distribution of two counter-rotating TE_{11} modes in a ferrite loaded circular waveguide. A ferrite rod magnetized along its axis presents an effective rf permeability of approximately two for the mode rotating in a negative screw sense with respect to the direction of magnetization. For the positive sense of rotation the effective rf permeability becomes very small and negligible energy is transmitted through the ferrite rod.

Unidirectional transmission characteristics were achieved by adding quarter wave plates before and after the ferrite rod and inserting an absorber into the ferrite. For the direction of propagation for which the quarter wave plate converts from a linear input to a positive circular rotation the positive wave tends to go around the ferrite with small loss. For the other direction of propagation the quarter wave plate converts the linear input wave to a negative wave which tends to concentrate in the ferrite and is absorbed.

Based on the principles described, an isolator was constructed which gives better than 30 db isolation over the range 8 to 11 kmc. The insertion loss is less than 2 db from 8 to 10.5 kmc and increases to 3 db at 11 kmc. The complete unit is 10½ inches long and weighs 2¼ pounds.

The main advantage of this isolator over present transverse field rectangular waveguide isolators and Faraday rotation isolators is its improved bandwidth. Other advantages are that the isolator is not sensitive to changes in magnetic field and it operates with a readily obtainable ferrite at low magnetic fields. Its vswr over the band is less than 1.2. The principle of this isolator is applicable to other frequency bands.

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Introduction

To KEEP abreast of current systems developments, both manufacturers and users of microwave components have felt the need for broadband microwave isolators. In attempts to make practical microwave isolators, ferrites have been heavily exploited in both circular and rectangular waveguide geometries. Effects of differential phase shift and differential resonance absorption for two directions of wave propagation are widely used as the basis for isolation in both waveguide geometries. In most reported cases, however, the bandwidth of these devices does not exceed ten per cent.